A Numerical Algorithm for Computing the Restricted Singular Value Decomposition of Matrix Triplets

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Submitted by Richard A. Brualdi

ABSTRACT

An algorithm is developed for computing the restricted singular value decomposition (RSVD) of general matrix triplets. It consists of three stages: we first show that orthogonal transformations can be constructed to extract a regular triplet from a given general matrix triplet; after preprocessing the regular triplet, we reduce the problem to computing the ordinary singular value decomposition of a product of three upper triangular matrices having the same number of dimensions; then we apply the implicit Kogbetliantz technique to this matrix product. Other structural indices of the RSVD can be obtained by computing the ranks of certain submatrices. Numerical examples are provided to illustrate the accuracy of the algorithm.

1. INTRODUCTION

The *restricted singular value decomposition* (RSVD) is a generalization of the ordinary singular value decomposition (OSVD)[1] for the case of three matrices with compatible dimensions: $A \in R^{m \times n}$, $B \in R^{m \times p}$, and $C \in R^{q \times n}$ [9]. The RSVD is useful in analyzing some theoretical problems such as the

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[1]The reader is invited to consult [2] for a standardized nomenclature for several generalizations of the OSVD.

655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

unitarily invariant norm minimization with rank constraints, the rank minimization of partitioned matrices, and the extended shorted operator problem [1]. It is the basis for the restricted total least squares problem [8] and for the regularization problem of regression analysis associated with the general Gauss-Markov linear model [10]. It also has application in canonical correlation analysis in statistics and signal processing [3]. For these and some other related problems there is a need for a numerical algorithm for computing the RSVD of a general matrix triplet.

The main emphasis of the algorithm proposed in this paper is on numerical accuracy. We will keep on using orthogonal transformations wherever possible so as to enhance the numerical stability of the algorithm. The paper is organized as follows: in Section 2, we give a brief account of the theory of the RSVD together with the concept of regular matrix triplets, which play an important role in the subsequent sections. We also point out the reasons why several constructive proofs of the RSVD in [9, 1] are not suitable for practical computation. In Section 3, we derive the main part of the algorithm: we show how we can only apply orthogonal transformations to the individual matrices A, B, and C to extract a regular subtriplet; in Section 4, we given an efficient method to preprocess the regular triplet, which will result in three upper triangular matrices of the same dimension; in Section 5, we apply an implicit Kogbetliantz technique to the product of three matrices and compare its numerical accuracy with that of the explicit method. In Section 6, we show how the other structural indices in the RSVD can be obtained by computing the ranks of certain submatrices of the transformed matrices A, B, and C. We also use nonsingular transformations to restore the three matrices to quasidiagonal forms (for the definition, see Theorem 1 below), which give the complete RSVD of the original matrix triplet (A, B, C). The whole procedure can be considered as another constructive proof of the RSVD in comparison with the proofs in [9, 1]. In Section 7 we report a numerical result.

NOTATION. Throughout the paper, matrices are denoted by capitals, and vectors by lowercase letters other than $i, j, k, l, m, n, p, q, r, s$, which are reserved for nonnegative integers. The symbol $R^{m\times n}$ represents the set of $m \times n$ real matrices. $\|\cdot\|$ is the spectrum norm, and $\|\cdot\|_F$ the Frobenius norm. The identity matrix of order j is denoted by I_j; we will omit the subscript when the dimension is clear from the context. A zero matrix is denoted by O with various dimensions. We also adopt the following convention for block matrices: whenever a dimension indicating integer in a block matrix is zero, the corresponding block row or column should be omitted, and all expressions and equations in which a block matrix of that block row or block column appears can be discarded.

The following abbreviations are used for matrix ranks:

$$r_A = \mathrm{rank}(A), \qquad r_B = \mathrm{rank}(B), \qquad r_C = \mathrm{rank}(C),$$

$$r_{AB} = \mathrm{rank}(A, B), \qquad r_{AC} = \mathrm{rank}\begin{pmatrix} A \\ C \end{pmatrix}, \qquad r_{ABC} = \mathrm{rank}\begin{pmatrix} A & B \\ C & O \end{pmatrix},$$

and the following nomenclature is used throughout the paper [2]:

OSVD: the ordinary singular value decomposition.
PSVD: the product singular value decomposition.
QSVD: the quotient singular value decomposition.
RSVD: the restricted singular value decomposition.

In this paper only the case of real matrices is discussed; extensions to the case of complex matrices are straightforward.

2. THE RESTRICTED SINGULAR VALUE DECOMPOSITION

In this section we briefly discuss the theory of the RSVD and introduce the concept of regular restricted singular values and regular matrix triplets. A more detailed analysis can be found in [9, 1]. The RSVD is a simultaneous factorization of three matrices with compatible dimensions to quasidiagonal forms. It can also be considered as the OSVD of the matrix A with respect to two semiinner products generated by the matrices B^T and C respectively.[2] The RSVD is described in the following theorem.

THEOREM 1 [9]. *Let $A \in R^{m \times n}$, $B \in R^{m \times p}$, and $C \in R^{q \times n}$. Then there exist nonsingular matrices $P \in R^{m \times m}$, $Q \in R^{n \times n}$ and orthogonal matrices $U \in R^{p \times p}$, $V \in R^{q \times q}$ such that*

$$P^{-1}AQ = \Sigma_A, \qquad P^{-1}BU = \Sigma_B, \qquad V^TCQ = \Sigma_C,$$

[2] The semiinner product x^TF^TFx is said to be generated by the matrix F.

where

$$\Sigma_A = \begin{array}{c c} \begin{array}{cccccc} j & k & l & r & s_1 & t_1 \end{array} & \\ \begin{pmatrix} I_j & O & O & O & O & O \\ O & I_k & O & O & O & O \\ O & O & I_l & O & O & O \\ O & O & O & S_A & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \end{pmatrix} & \begin{array}{c} j \\ k \\ l \\ r \\ s_2 \\ t_2 \end{array} \end{array},$$

$$\Sigma_B = \begin{array}{c c} \begin{array}{cccc} j & p-j-r-s_2 & r & s_2 \end{array} & \\ \begin{pmatrix} I_j & O & O & O \\ O & O & O & O \\ O & O & S_B & O \\ O & O & O & I_{s_2} \\ O & O & O & O \end{pmatrix} & \begin{array}{c} j \\ k+l \\ r \\ s_2 \\ t_2 \end{array} \end{array},$$

$$\Sigma_C = \begin{array}{c c} \begin{array}{ccccc} j+k & l & r & s_1 & t_1 \end{array} & \\ \begin{pmatrix} O & O & O & O & O \\ O & I_l & O & O & O \\ O & O & S_C & O & O \\ O & O & O & I_{s_1} & O \end{pmatrix} & \begin{array}{c} q-l-r-s_1 \\ l \\ r \\ s_1 \end{array} \end{array},$$

and $S_A = \mathrm{diag}(\alpha_i)$, $S_B = \mathrm{diag}(\beta_i)$, *and* $S_C = \mathrm{diag}(\gamma_i)$ *are diagonal matrices with positive diagonal elements. The integer indices can be expressed as follows*:

$$j = r_{AC} + r_B - r_{ABC},$$

$$k = r_{ABC} - r_B - r_C,$$

$$l = r_{AB} + r_C - r_{ABC},$$

$$r = r_{ABC} + r_A - r_{AB} - r_{AC},$$

$$s_1 = r_{AC} - r_A,$$

$$s_2 = r_{AB} - r_A,$$

$$t_1 = n - r_{AC},$$

$$t_2 = m - r_{AB}.$$

In order to discuss the properties of the RSVD we first give the following definition.

DEFINITION 1. We call the following triplets of numbers the *restricted singular value triplets* of A, B, and C:

$$\alpha_i = 1, \quad \beta_i = 1, \quad \gamma_i = 0, \qquad i = 1,\dots,j,$$

$$\alpha_i = 1, \quad \beta_i = 0, \quad \gamma_i = 0, \qquad i = j+1,\dots,j+k,$$

$$\alpha_i = 1, \quad \beta_i = 0, \quad \gamma_i = 1, \qquad i = j+k+1,\dots,s,$$

$$\alpha_i, \beta_i, \gamma_i \text{ as in } S_A,\ S_B, \text{ and } S_C, \qquad i = s+1,\dots s+r,$$

$$\alpha_i = 0, \quad \beta_i = 1, \quad \gamma_i = 1, \qquad i = s+r+1,\dots,s+r+\min(s_1, s_2),$$

where we have used $s = j + k + l$. Furthermore $\sigma_i = \alpha_i / \beta_i \gamma_i$, $i = 1,\dots,s + r + \min(s_1, s_2)$, are called the *restricted singular values* of A, B, and C. We note that σ_i, $i = 1,\dots,s$, are infinite.

An important property of the RSVD is the following rank characterization of the restricted singular values.

THEOREM 2 [9]. *The restricted singular values of the matrix triplet A, B, and C can be characterized as*

$$\sigma_i = \min_{D \in R^{p \times q}} \{\|D\| | \operatorname{rank}(A + BDC) \leqslant i - 1\}, \qquad i = 1,\dots,s+r+\min(s_1, s_2).$$

$\sigma_i = \infty$ corresponds to the situation that we cannot find any matrix D to make the rank of $A + BDC$ smaller than or equal to $i - 1$.

Now we introduce the concepts of regular restricted singular value triplets, regular restricted singular values, and regular matrix triplets.

DEFINITION 2. The triplets of numbers

$$\alpha_i, \beta_i, \gamma_i \text{ as in } S_A,\ S_B, \text{ and } S_C, \qquad i = s+1,\dots s+r,$$

$$\alpha_i = 0, \quad \beta_i = 1, \quad \gamma_i = 1, \qquad i = s+r+1,\dots,s+r+\min(s_1, s_2),$$

are called the *regular restricted singular triplets of* A, B, and C. The corresponding restricted singular values are called the *regular restricted singular values*. A matrix triplet (A, B, C) is called *regular* if it only has regular restricted singular value triplets.

Using Theorem 1, we can verify the following characterization of regular matrix triplets.

COROLLARY 1. *A matrix triplet* (A, B, C) *is regular if and only if* B *is of full row rank and* C *is of full column rank. Specifically,* (A, B, C) *is regular if* B *and* C *are nonsingular.*

In the next section it will be shown we can use orthogonal transformations to extract a regular triplet from a general matrix triplet.

As mentioned before, two constructive proofs of the RSVD are given in [9, 1]: one is based on the QSVD and OSVD, and the other is based on the PSVD and OSVD. One might have thought of using the ideas in the constructive proofs to form a numerical algorithm for computing the RSVD. However, we point out that in both approaches, nonsingular (not necessarily orthogonal) transformations are used to scale nonsingular matrices to identity matrices, and to eliminate certain submatrices in some *intermediate* steps during the decomposition. This will cause numerical instability if the underlying nonsingular matrices or the pivoting matrices are ill conditioned. Another possible way of computing the RSVD is to consider the corresponding eigenstructure problem of the symmetric matrix pencil

$$\begin{pmatrix} O & A^T \\ A & O \end{pmatrix} x = \lambda \begin{pmatrix} BB^T & O \\ O & C^TC \end{pmatrix} x,$$

which is associated with the RSVD of (A, B, C) [9], and use the well-established algorithms for computing the eigenstructure of matrix pencils [7]. The drawbacks of this method are twofold: first, forming the cross-product matrices will cause loss of numerical accuracy; secondly, the dimension of the problem is doubled, in conflict with the good numerical practice of applying orthogonal transformations to A, B, and C separately where possible. The method to be described in this paper applies orthogonal transformations to the individual matrices A, B, and C separately to compute the restricted singular triplets. Nonsingular transformations are used only when the complete RSVD is required.

3. EXTRACTING A REGULAR TRIPLET FROM A GENERAL MATRIX TRIPLET

In this section we will develop a method to extract a regular triplet from a general given matrix triplet (A, B, C). The restricted singular triplets of the regular triplet are exactly the *regular* restricted singular triplets of the original matrix triplet (A, B, C). Our main tool is applying orthogonal transformations to compress the rows or columns of certain matrices. We summarize the compression procedures in the following lemma.

LEMMA 1. *Let $A \in R^{m\times n}$. Then there exist orthogonal matrices $U \in R^{m\times m}$ and $V \in R^{n\times n}$ such that*

$$U^TA = \begin{pmatrix} A_r \\ O \end{pmatrix}, \qquad AV = (A_c, O),$$

where A_r (A_c) is of full row (column) rank.

Proof. Let the OSVD of A be

$$U^TAV = \Sigma = \begin{pmatrix} \Sigma_r & O \\ O & O \end{pmatrix},$$

where $\Sigma_r \in R^{r\times r}$ is diagonal with positive diagonal elements. Let $U = [U_1, U_2]$, $V = [V_1, V_2]$ be partitioned compatibly with the partitioning of Σ. Then the result follows by letting $A_r = \Sigma_r V_1^T$ and $A_c = U_1\Sigma_r$. ∎

We call the above procedures the *row compression* and the *column compression* of the matrix. They constitute the method we used in our implementation. We mention here that instead of using the OSVD, the QR decomposition with column pivoting or the rank revealing QR decomposition can also be used to compress the rows or columns of a matrix [4].

Our extraction stage consists of five steps. The transformation from step k to step $k+1$ is of the following form:

$$\begin{pmatrix} A^{(k+1)} & B^{(k+1)} \\ C^{(k+1)} & O \end{pmatrix} = \begin{pmatrix} \left(U_1^{(k)}\right)^T A^{(k)}\left(U_2^{(k)}\right) & \left(U_1^{(k)}\right)^T B^{(k)} V_1^{(k)} \\ \left(V_2^{(k)}\right)^T C^{(k)}\left(U_2^{(k)}\right) & O \end{pmatrix}$$

where $U_i^{(k)}$, $i = 1, 2$, and $V_i^{(k)}$, $i = 1, 2$, are orthogonal, and where $A^{(k)}$, $B^{(k)}$, and $C^{(k)}$ are the transformed A, B, and C at step k with initial values

$$A^{(0)} = A, \qquad B^{(0)} = B, \qquad C^{(0)} = C.$$

A little more explanation about the above procedure is in order here: whenever we apply orthogonal transformations to compress the rows of some of the submatrices of $A^{(k)}$, we should apply the same transformations to the rows of $B^{(k)}$. The same is true when we apply orthogonal transformations to compress the columns of $A^{(k)}$: we should apply the same transformations to the columns of $C^{(k)}$; the row transformations of $C^{(k)}$ and the column transformations of $B^{(k)}$ can be applied individually. We will write the transformed A, B, and C in block matrix form. For example $A_{12}^{(k)}$ will be the $(1,2)$ submatrix of the transformed A at stage k. All the matrices are compatibly partitioned.

THE EXTRACTION ALGORITHM.

Step 1. Compress the columns of $C^{(0)}$; we obtain

$$\begin{pmatrix} (A_{11}^{(1)}, A_{12}^{(1)}) & B^{(1)} \\ (C_1^{(1)}, O) & O \end{pmatrix},$$

where $C_1^{(1)}$ is of full column rank.

Step 2. Compress the rows of $A_{12}^{(1)}$; we obtain

$$\begin{pmatrix} \begin{pmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ A_{21}^{(2)} & O \end{pmatrix} & \begin{pmatrix} B_1^{(2)} \\ B_2^{(2)} \end{pmatrix} \\ (C_1^{(2)}, O) & O \end{pmatrix},$$

where $A_{12}^{(2)}$ is of full row rank, and $C_1^{(2)} = C_1^{(1)}$.

Step 3. Compress the rows of $B_2^{(2)}$; we obtain

$$\begin{pmatrix} \begin{pmatrix} A_{11}^{(3)} & A_{12}^{(3)} \\ A_{21}^{(3)} & O \\ A_{31}^{(3)} & O \end{pmatrix} & \begin{pmatrix} B_1^{(3)} \\ B_2^{(3)} \\ O \end{pmatrix} \\ (C_1^{(3)}, O) & O \end{pmatrix},$$

where $B_2^{(3)}$ is of full row rank, and $A_{11}^{(3)} = A_{11}^{(2)}$, $A_{12}^{(3)} = A_{12}^{(2)}$, $B_2^{(3)} = B_2^{(2)}$, and $C_1^{(3)} = C_1^{(2)}$.

Step 4. Compress the columns of $A_{31}^{(3)}$; we obtain

$$\left(\begin{array}{cc}\begin{pmatrix} A_{11}^{(4)} & A_{12}^{(4)} & A_{13}^{(4)} \\ A_{21}^{(4)} & A_{22}^{(4)} & O \\ A_{31}^{(4)} & O & O \end{pmatrix} & \begin{pmatrix} B_1^{(4)} \\ B_2^{(4)} \\ O \end{pmatrix} \\ \left(C_1^{(4)}, C_2^{(4)}, O\right) & O \end{array}\right),$$

where $A_{31}^{(4)}$ is of full column rank, and $A_{13}^{(4)} = A_{12}^{(3)}$, $B_1^{(4)} = B_1^{(3)}$, $B_2^{(4)} = B_2^{(3)}$.

Step 5. Compress the columns of $B_2^{(4)}$ and compress the rows of $C_2^{(4)}$; we obtain

$$\left(\begin{array}{cc}\begin{pmatrix} A_{11}^{(5)} & A_{12}^{(5)} & A_{13}^{(5)} \\ A_{21}^{(5)} & A_{22}^{(5)} & O \\ A_{31}^{(5)} & O & O \end{pmatrix} & \begin{pmatrix} B_{11}^{(5)} & B_{12}^{(5)} \\ B_{21}^{(5)} & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} C_{11}^{(5)} & C_{12}^{(5)} & O \\ C_{21}^{(5)} & O & O \end{pmatrix} & O \end{array}\right),$$

where $B_{21}^{(4)}$ and $C_{12}^{(5)}$ are nonsingular, and all the block submatrices of $A^{(5)}$ remain the same, i.e. $A^{(5)} = A^{(4)}$.

THEOREM 3. *All restricted singular triplets of* $(A_{22}^{(5)}, B_{21}^{(5)}, C_{12}^{(5)})$ *are regular, and they coincide with the regular restricted singular triplets of the original matrix triplet* (A, B, C).

Proof. Since $B_2^{(4)}$ is of full row rank and $C_2^{(4)}$ is of full column rank, $B_{21}^{(5)}$ and $C_{12}^{(5)}$ are nonsingular. Hence $(A_{22}^{(5)}, B_{21}^{(5)}, C_{12}^{(5)})$ is a regular matrix triplet and only has regular restricted singular triplets. Now consider the block structure of $A^{(5)} + B^{(5)}D^{(5)}C^{(5)}$, where $D^{(5)}$ is partitioned compatibly with the block partitioning of $B^{(5)}$ and $C^{(5)}$; then

$$A^{(5)} + B^{(5)}D^{(5)}C^{(5)} = \begin{pmatrix} X_{11} & X_{12} & A_{13}^{(5)} \\ X_{21} & A_{22}^{(5)} + B_{21}^{(5)}D_{11}^{(5)}C_{12}^{(5)} & O \\ A_{31}^{(5)} & O & O \end{pmatrix},$$

where X_{11}, X_{12}, and X_{21} are possibly nonzero submatrices of $A^{(5)} + B^{(5)}D^{(5)}C^{(5)}$. Since in step 1 to step 5 of the extraction algorithm all the transformations are orthogonal and hence nonsingular, and $A_{13}^{(5)}$ has full row rank and $A_{31}^{(5)}$ has full column rank, we obtain

$$\begin{aligned}
\operatorname{rank}(A + BDC) &= \operatorname{rank}(A^{(5)} + B^{(5)}D^{(5)}C^{(5)}) \\
&= \operatorname{rank}\begin{pmatrix} X_{11} & X_{12} & A_{13}^{(5)} \\ X_{21} & A_{22}^{(5)} + B_{21}^{(5)}D_{11}^{(5)}C_{12}^{(5)} & O \\ A_{31}^{(5)} & O & O \end{pmatrix} \\
&= \operatorname{rank}(A_{13}^{(5)}) + \operatorname{rank}(A_{31}^{(5)}) + \operatorname{rank}(A_{22}^{(5)} + B_{21}^{(5)}D_{11}^{(5)}C_{12}^{(5)}).
\end{aligned}$$

The result follows from the rank characterization of the restricted singular values. ■

4. PREPROCESSING THE REGULAR TRIPLET

In this section we make some preparations for the next section where we will apply an implicit Kogbetliantz technique to the product of three matrices. We apply orthogonal transformations to the regular triplet $(A_{22}^{(5)}, B_{21}^{(5)}, C_{12}^{(5)})$, which will result in three upper triangular matrices of the same dimension.

We start with a regular triplet (A, B, C), where $A \in R^{m\times n}$, $B \in R^{m\times m}$, and $C \in R^{n\times n}$ and we also assume that B and C are nonsingular. Using Theorem 2, it can be readily checked that the restricted singular values of (A, B, C) are just the singular values of $B^{-1}AC^{-1}$. So from now on we will concentrate on the product $B^{-1}AC^{-1}$. Obviously a straightforward way to compute the singular values of $B^{-1}AC^{-1}$ is the following:

THE EXPLICIT ALGORITHM.

Solve the linear equation $BY = A$ for Y.
Solve the linear equation $XC = Y$ for X.
Compute the OSVD of X.

As is well known, this approach will lead to large numerical error when the matrices B and C are ill conditioned, so generally it is not recommended. In the sequel we will stick to applying orthogonal transformations to the individual matrices A, B, and C and not explicitly forming the product. For the purpose of the preprocessing procedure we distinguish two cases:

1. $m \geqslant n$ We start with C, and let its QR decomposition be

$$C = Q_C R_C, \tag{1}$$

where Q_C is orthogonal and R_C is upper triangular. We write the QR decomposition of A in the following way:

$$A = Q_A \begin{pmatrix} R_A \\ O \end{pmatrix}, \tag{2}$$

where Q_A is orthogonal and $R_A \in R^{n\times n}$ is upper triangular. Let $\tilde{B} = Q_A^T B$, and apply an orthogonal transformation to the right of $\tilde{B}$ to make it upper triangular such that

$$\tilde{B} = \begin{pmatrix} R_B & B_{12} \\ O & B_{22} \end{pmatrix} (Q_B)^T, \tag{3}$$

where Q_B is orthogonal and $R_B \in R^{n\times n}$ is upper triangular. Combining the above, we obtain

$$Q_B^T (B^{-1}AC^{-1}) Q_C = \begin{pmatrix} R_B^{-1} R_A R_C^{-1} \\ O \end{pmatrix}. \tag{4}$$

Therefore the problem of finding the singular values of $B^{-1}AC^{-1}$ reduces to that of finding the singular values of $R_B^{-1}R_AR_C^{-1}$. We observe that R_A, R_B, and R_C are upper triangular and have the same dimension.

2. $m < n$. Similarly to the above case, we can find Q_A, Q_B, and Q_C orthogonal such that

$$Q_B^T (B^{-1}AC^{-1}) Q_C = (R_B^{-1} R_A R_C^{-1}, O),$$

where $R_A \in R^{m\times m}$, $R_B \in R^{m\times m}$, and $R_C \in R^{m\times m}$ are upper triangular.

Before we proceed to the next section, we will give an alternative for the preprocessing procedure. At step 4 of the above extraction algorithm, we end up with the triplet $(A_{22}^{(4)}, B_2^{(4)}, C_2^{(4)})$. It is interesting to observe that if the QR decomposition is used to compress the rows and columns of $B_2^{(4)}$ and $C_2^{(4)}$, we obtain two matrices $B_{21}^{(4)}$ and $C_{12}^{(5)}$ which are upper triangular. If we directly apply the preprocessing algorithm given above, then the upper triangular form of one of the matrices will be destroyed. We observe that in Equations (2) and (3), what we actually do is find two orthogonal matrices Q_A and Q_B such that Q_A^TA and $Q_A^TBQ_B$ are upper triangular. We will show that we can combine these two steps in Equations (2) and (3) into a single step, so that we transform A to upper triangular form while at the same time we preserve the triangular form of B. The idea can be fully illustrated by using a low dimension example. We assume $m = 3$ and $n = 2$, i.e., B is a 3×3 upper triangular matrix and A is a 3×2 general matrix. As used conventionally,

"$\times$" represents possible nonzero entries of a matrix; "$\otimes$" represents the entry to be zeroed out at the current step; a blank space represents a zero entry. So initially we have

$$B = \begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{pmatrix}, \qquad A = \begin{pmatrix} \times & \times \\ \times & \times \\ \times & \times \end{pmatrix}.$$

We use Givens rotations to introduce zero entries in a selective manner; throughout, "$\rightarrow$" indicates the two rows or columns involved in the Givens rotation:

$$\begin{array}{c} \\ \rightarrow \\ \rightarrow \end{array} \begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{pmatrix} \qquad \begin{pmatrix} \times & \times \\ \times & \times \\ \otimes & \times \end{pmatrix}$$

$$\begin{array}{ccc} & \downarrow & \downarrow \end{array}$$
$$\begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & \otimes & \times \end{pmatrix} \qquad \begin{pmatrix} \times & \times \\ \times & \times \\ & \times \end{pmatrix}$$

$$\begin{array}{c} \rightarrow \\ \rightarrow \\ \\ \end{array} \begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{pmatrix} \qquad \begin{pmatrix} \times & \times \\ \otimes & \times \\ & \times \end{pmatrix}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \end{array}$$
$$\begin{pmatrix} \times & \times & \times \\ \otimes & \times & \times \\ & \times & \times \end{pmatrix} \qquad \begin{pmatrix} \times & \times \\ & \times \\ & \times \end{pmatrix}$$

$$\begin{array}{c} \\ \rightarrow \\ \rightarrow \end{array} \begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{pmatrix} \qquad \begin{pmatrix} \times & \times \\ & \times \\ & \otimes \end{pmatrix}$$

$$\begin{array}{ccc} & \downarrow & \downarrow \end{array}$$
$$\begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & \otimes & \times \end{pmatrix} \qquad \begin{pmatrix} \times & \times \\ & \times \end{pmatrix}$$

$$\begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{pmatrix} \qquad \begin{pmatrix} \times & \times \\ & \times \end{pmatrix}$$

A noteworthy remark is in order here: whenever we apply a rowwise Givens rotation, we apply it both to the first and to the second matrix, while we apply columnwise Givens rotations only to the first matrix. Using the Givens rotation version in [4], the operation counts of the above procedure are roughly $2(2m+n)n$ flops.

5. IMPLICIT KOGBETLIANTZ TECHNIQUE APPLIED TO THE PRODUCT OF THREE MATRICES

In this section we apply an implicit Kogbetliantz technique to compute the OSVD of $B^{-1}AC^{-1}$, where $A \in R^{n\times n}$, $B \in R^{n\times n}$, and $C \in R^{n\times n}$ are upper triangular matrices. The main emphasis is that we will not explicitly form the product. The implicit Kogbetliantz technique has been applied to the computation of the PSVD [5] and QSVD [6]. We will only present the core algorithm and suggest that the reader consult the abovementioned two papers and references therein for issues like ordering schemes, convergence analysis, and parallel (systolic) implementation.

Let $E = B^{-1}AC^{-1}$ and $E=(e_{ij})$, $A=(a_{ij})$, $B=(b_{ij})$, $C=(c_{ij})$. Since A, B, and C are upper triangular, we have

$$\begin{pmatrix} e_{ii} & e_{ii+1} \\ 0 & e_{i+1i+1} \end{pmatrix} = \begin{pmatrix} b_{ii} & b_{ii+1} \\ 0 & b_{i+1i+1} \end{pmatrix}^{-1} \begin{pmatrix} a_{ii} & a_{ii+1} \\ 0 & a_{i+1i+1} \end{pmatrix} \times \begin{pmatrix} c_{ii} & c_{ii+1} \\ 0 & c_{i+1i+1} \end{pmatrix}^{-1}, \qquad i=1,\dots,n-1. \tag{5}$$

Since the adjoint of a matrix is just a scalar multiple of its inverse, we can replace the inverses in (5) by their adjoints. This idea was first introduced in [6]. By doing so, the result can also cover the cases where B and C are singular. So the products are of the following form:

$$\begin{pmatrix} f_{ii} & f_{ii+1} \\ 0 & f_{i+1i+1} \end{pmatrix} = \operatorname{adj}\begin{pmatrix} b_{ii} & b_{ii+1} \\ 0 & b_{i+1i+1} \end{pmatrix} \begin{pmatrix} a_{ii} & a_{ii+1} \\ 0 & a_{i+1i+1} \end{pmatrix} \times \operatorname{adj}\begin{pmatrix} c_{ii} & c_{ii+1} \\ 0 & c_{i+1i+1} \end{pmatrix}, \qquad i=1,\dots,n-1. \tag{6}$$

To make the presentation concise, we will concentrate on the following product:

$$\begin{pmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{pmatrix} = \operatorname{adj}\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\operatorname{adj}\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix};$$

the general case can be considered similarly. The main issue is to compute the 2×2 OSVD of the above product, while preserving the upper triangular form of the individual matrices. The result is given in the following theorem.

THEOREM 4. *Let J_1 and J_2 be the combination of rotations and permutations such that*

$$J_1^T\begin{pmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{pmatrix}J_2 = \begin{pmatrix} \hat{f}_{11} & 0 \\ 0 & \hat{f}_{22} \end{pmatrix}. \tag{7}$$

Then we can construct 2×2 orthogonal matrices J_3 and J_4 such that

$$J_3^T\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}J_1, \qquad J_3^T\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}J_4, \qquad J_2^T\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix}J_4$$

are 2×2 upper triangular matrices.

Proof. Let

$$\bar{B} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}J_1 = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{pmatrix},$$

$$\bar{C} = J_2^T\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix} = \begin{pmatrix} \bar{c}_{11} & \bar{c}_{12} \\ \bar{c}_{21} & \bar{c}_{22} \end{pmatrix}.$$

In the following we distinguish three cases:

1. If $(\bar{b}_{11})^2+(\bar{b}_{21})^2 \neq 0$ and $(\bar{c}_{21})^2+(\bar{c}_{22})^2 \neq 0$, then choose J_3 and J_4 such that

$$J_3^T\bar{B} = \begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} \\ 0 & \hat{b}_{22} \end{pmatrix}, \qquad \bar{C}J_4 = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} \\ 0 & \hat{c}_{22} \end{pmatrix}.$$

It can be readily checked that $\hat{b}_{11} \neq 0$ and $\hat{c}_{22} \neq 0$. Let

$$J_3^T \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} J_4 = \begin{pmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{pmatrix}.$$

From (6) we have

$$\begin{pmatrix} \hat{f}_{11} & 0 \\ 0 & \hat{f}_{22} \end{pmatrix} = \operatorname{adj}\begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} \\ 0 & \hat{b}_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{pmatrix} \operatorname{adj}\begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} \\ 0 & \hat{c}_{22} \end{pmatrix}.$$

Comparing the (2,1) elements of the above equation, we obtain $\hat{a}_{21} = 0$.

2. If $(\bar{b}_{11})^2 + (\bar{b}_{21})^2 = 0$, then choose J_4 such that

$$\bar{C}J_4 = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} \\ 0 & \hat{c}_{22} \end{pmatrix},$$

and choose J_3 such that

$$J_3^T \left[\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} J_4 \right] = \begin{pmatrix} \hat{a}_{11} & \hat{a}_{12} \\ 0 & \hat{a}_{22} \end{pmatrix}.$$

Then

$$J_3\bar{B} = \begin{pmatrix} 0 & \hat{b}_{12} \\ 0 & \hat{b}_{22} \end{pmatrix}.$$

3. If $(\bar{c}_{21})^2 + (\bar{c}_{22})^2 = 0$, then choose J_3 such that

$$J_3\bar{B} = \begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} \\ 0 & \hat{b}_{22} \end{pmatrix},$$

and choose J_4 such that

$$\left[J_3 \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \right] J_4 = \begin{pmatrix} \hat{a}_{11} & \hat{a}_{12} \\ 0 & \hat{a}_{22} \end{pmatrix}.$$

Then

$$\tilde{C}J_4 = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} \\ 0 & 0 \end{pmatrix}. \qquad \blacksquare$$

An alternative method was also independently proposed in [3]. Some other methods are still under investigation, and careful error analysis and numerical experiments need to be carried out in order to compare the numerical performance and stability of all these methods. But these are beyond the scope of the present paper.

We use a numerical example to demonstrate that the implicit method has better numerical stability than the explicit method, especially when the product $B^{-1}AC^{-1}$ has very small singular values. A similar phenomenon was also observed in [5]. We let

$$A = P^{-1}\Sigma_A Q, \qquad B = P^{-1}\Sigma_B U, \qquad C = V^T \Sigma_C Q,$$

where P and Q are nonsingular and U and V are orthogonal. The singular values of $B^{-1}AC^{-1}$ are given by the diagonal elements of $(\Sigma_B)^{-1}\Sigma_A(\Sigma_C)^{-1}$. A typical example with $\Sigma_B(i,i) = \Sigma_C(i,i) = 10^{i/4}$ and $\Sigma_A(i,i) = 10^{-i/2}$, where $i = 1, \ldots, 20$, is given in Figure 1. All the computation was done on Sun 3/50 workstation using Matlab version 3.5c. The singular values are

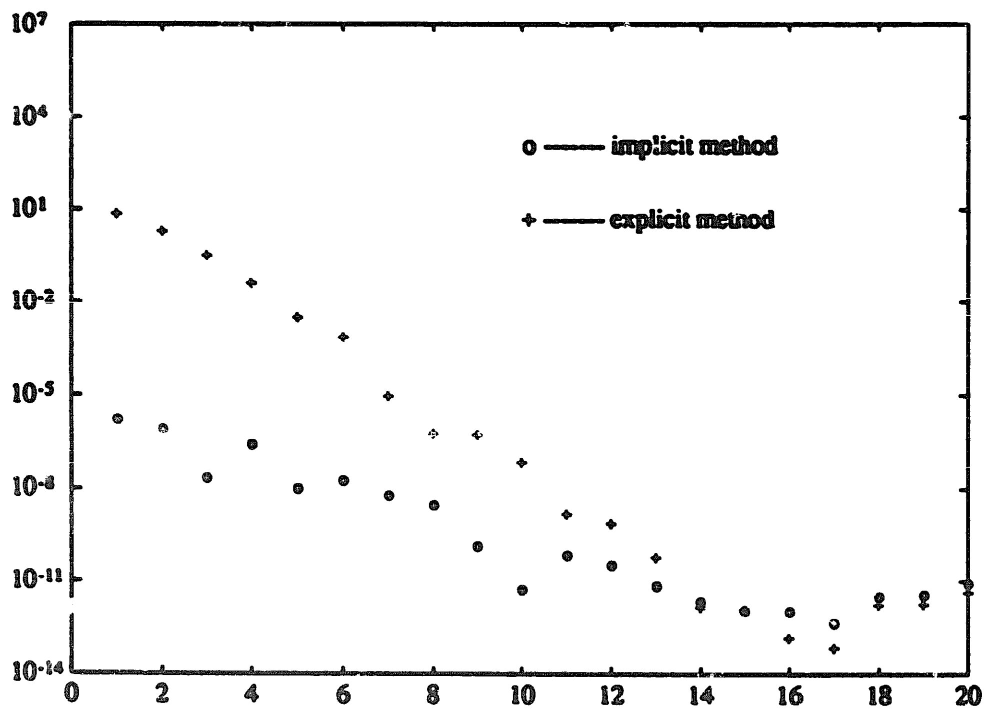

FIG. 1. Comparison of relative errors of the computed singular values.

arranged in nondecreasing order. The singular values computed by the explicit method show a steady decline in accuracy, the smallest singular values having no significant digits of accuracy. The implicit method, although it suffers some loss of accuracy, still provides four digits of accuracy. (See Figure 1.)

6. COMPUTING THE OTHER RESTRICTED SINGULAR VALUE TRIPLETS AND THE COMPLETE RSVD

In this section we show that all the other restricted singular value triplets can be obtained by computing the ranks of certain submatrices of $A^{(5)}$, $B^{(5)}$, and $C^{(5)}$ (see step 5 of the extraction algorithm). We will also use nonsingular transformations to restore $A^{(5)}$, $B^{(5)}$, and $C^{(5)}$ to quasidiagonal form and thus obtain the complete RSVD.

THEOREM 5. *Using the notation in step 5 of the extraction algorithm, let*

$$\operatorname{rank}\left(B_{12}^{(5)}\right) = \tilde{r}_B, \qquad \operatorname{rank}\left(C_{21}^{(5)}\right) = \tilde{r}_C.$$

Furthermore assume that $A_{22}^{(5)} \in R^{m_2 \times n_2}$, $B_{12}^{(5)} \in R^{m_1 \times p_1}$, $C_{21}^{(5)} \in R^{q_1 \times n_1}$. *Then the other restricted singular triplets of* (A, B, C) *are*

(1) $\tilde{r}_B$ *restricted singular value triplets of the form* $(1,1,0)$,
(2) $\tilde{r}_C$ *restricted singular value triplets of the form* $(1,0,1)$,
(3) $m_1 - \tilde{r}_B + n_1 - \tilde{r}_C$ *restricted singular value triplets of the form* $(1,0,0)$.

The integer indices are

$$j = \tilde{r}_B,$$

$$k = m_1 - \tilde{r}_B + n_1 - \tilde{r}_C,$$

$$l = \tilde{r}_C,$$

$$r = \textit{number of nonzero singular values of } \left(B_{21}^{(5)}\right)^{-1} A_{22}^{(5)} \left(C_{12}^{(5)}\right)^{-1},$$

$$s_1 = n_2 - r,$$

$$s_2 = m_2 - r.$$

Proof. We will transform the matrices $A^{(5)}$, $B^{(5)}$, and $C^{(5)}$ to quasidiagonal form; the results of the theorem follow as by-products. The reduction procedure consists of four steps. The transformation of each step is of the following form:

$$\begin{pmatrix} A^{(k+1)} & B^{(k+1)} \\ C^{(k+1)} & O \end{pmatrix} = \begin{pmatrix} (P^{(k)})^{-1} A^{(k)} Q^{(k)} & (P^{(k)})^{-1} B^{(k)} U^{(k)} \\ (V^{(k)})^T C^{(k)} Q^{(k)} & O \end{pmatrix},$$

where $P^{(k)}$ and $Q^{(k)}$ are nonsingular and $U^{(k)}$ and $V^{(k)}$ are orthogonal. In each step we will specify the $P^{(k)}$, $Q^{(k)}$, $U^{(k)}$, and $V^{(k)}$ and the resulting $A^{(k+1)}$, $B^{(k+1)}$, and $C^{(k+1)}$. We continue step 5 in the extraction algorithm and start with the matrices $A^{(5)}$, $B^{(5)}$, and $C^{(5)}$.

Step 1. After preprocessing the regular triplet and applying the Kogbetliantz algorithm, we found orthogonal matrices $\tilde{U}_6$ and $\tilde{V}_6$ such that

$$(\tilde{U}_6)^T (B_{21}^{(5)})^{-1} A_{22}^{(5)} (C_{12}^{(5)})^{-1} \tilde{V}_6 = \Sigma = \begin{pmatrix} \Sigma_r & O \\ O & O \end{pmatrix},$$

where Σ_r is diagonal with positive diagonal entries. Let

$$(P^{(6)})^{-1} = \operatorname{diag}\left(I, (B_{21}^{(5)})^{-1}, I\right), \qquad Q^{(6)} = \operatorname{diag}\left(I, (C_{12}^{(5)})^{-1}, I\right),$$

$$U^{(6)} = \operatorname{diag}\left(I, \tilde{U}_6, I\right), \qquad V^{(6)} = \operatorname{diag}\left(I, \tilde{V}_6, I\right);$$

then we have

$$\begin{pmatrix} \begin{pmatrix} A_{11}^{(6)} & A_{12}^{(6)} & A_{13}^{(6)} \\ A_{21}^{(6)} & \Sigma & O \\ A_{31}^{(6)} & O & O \end{pmatrix} & \begin{pmatrix} B_{11}^{(6)} & B_{12}^{(6)} \\ I_{m_2} & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} C_{11}^{(6)} & I_{n_2} & O \\ C_{21}^{(6)} & O & O \end{pmatrix} & O \end{pmatrix}.$$

Step 2. First we can use I_{m_2} as pivot to eliminate $B_{11}^{(6)}$, and I_{n_2} as pivot to eliminate $C_{11}^{(6)}$. Furthermore, since $A_{13}^{(6)}$ has full row rank and $A_{31}^{(6)}$ has full column rank, we can find nonsingular matrices $P_7^{(1)}$, $P_7^{(2)}$,

$Q_7^{(1)}$, and $Q_7^{(2)}$ such that

$$\left(P_7^{(1)}\right)^{-1}A_{13}^{(6)}Q_7^{(2)}=\left(I_{m_1},O\right),\qquad \left(P_7^{(2)}\right)^{-1}A_{31}^{(6)}Q_7^{(1)}=\begin{pmatrix} I_{n_1} \\ O \end{pmatrix}.$$

Let

$$\left(P^{(7)}\right)^{-1}=\begin{pmatrix} \left(P_7^{(1)}\right)^{-1} & O & O \\ O & I & O \\ O & O & \left(P_7^{(2)}\right)^{-1} \end{pmatrix}\begin{pmatrix} I & -B_{11}^{(6)} & O \\ O & I & O \\ O & O & I \end{pmatrix},$$

$$Q^{(7)}=\begin{pmatrix} I & O & O \\ -C_{11}^{(6)} & I & O \\ O & O & I \end{pmatrix}\begin{pmatrix} Q_7^{(1)} & O & O \\ O & I & O \\ O & O & Q_7^{(2)} \end{pmatrix};$$

thc . we have

$$\begin{pmatrix} \begin{pmatrix} A_{11}^{(7)} & A_{12}^{(7)} & \left(I_{m_1},O\right) \\ A_{21}^{(7)} & \Sigma & O \\ \begin{pmatrix} I_{n_1} \\ O \end{pmatrix} & O & O \end{pmatrix} & \begin{pmatrix} O & B_{12}^{(7)} \\ I_{m_2} & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & I_{n_2} & O \\ C_{21}^{(7)} & O & O \end{pmatrix} & O \end{pmatrix}.$$

Step 3. We can use I_{m_1} and I_{n_1} as pivots to eliminate $A_{11}^{(7)}$, $A_{12}^{(7)}$, and $A_{21}^{(7)}$. Let

$$\left(P^{(8)}\right)^{-1}=\begin{pmatrix} I & O & -\frac{1}{2}A_{11}^{(7)} & O \\ O & I & -A_{21}^{(7)} & O \\ O & O & I & O \\ O & O & O & I \end{pmatrix},\qquad Q^{(8)}=\begin{pmatrix} I & O & O & O \\ O & I & O & O \\ -\frac{1}{2}A_{11}^{(7)} & A_{12}^{(7)} & I & O \\ O & O & O & I \end{pmatrix};$$

then we have

$$\left(\begin{array}{cc}\begin{pmatrix} O & O & (I_{m_1},O) \\ O & \Sigma & O \\ \begin{pmatrix} I_{n_1} \\ O \end{pmatrix} & O & O \end{pmatrix} & \begin{pmatrix} O & B_{12}^{(8)} \\ I_{m_2} & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & I_{n_2} & O \\ C_{21}^{(8)} & O & O \end{pmatrix} & O \end{array}\right).$$

Step 4. Let the OSVD of $B_{12}^{(8)}$ and $C_{21}^{(8)}$ be

$$\left(U_9^{(1)}\right)^T B_{12}^{(8)} V_9^{(1)} = \begin{pmatrix} \Sigma_b & O \\ O & O \end{pmatrix}, \qquad \left(U_9^{(2)}\right)^T C_{21}^{(8)} V_9^{(2)} = \begin{pmatrix} \Sigma_c & O \\ O & O \end{pmatrix}$$

where $\Sigma_b \in R^{\tilde{r}_B \times \tilde{r}_B}$ and $\Sigma_c \in R^{\tilde{r}_C \times \tilde{r}_C}$ are diagonal matrices with positive diagonal entries. Let

$$\left(P^{(9)}\right)^{-1} = \begin{pmatrix} \Sigma_b^{-1}\left(U_9^{(1)}\right)^T & O & O & O \\ O & I & O & O \\ O & O & \left(V_9^{(2)}\Sigma_c^{-1}\right)^{-1} & O \\ O & O & O & I \end{pmatrix},$$

$$Q^{(9)} = \begin{pmatrix} V_9^{(2)}\Sigma_c^{-1} & O & O & O \\ O & I & O & O \\ O & O & \left[\Sigma_b^{-1}\left(U_9^{(1)}\right)^T\right]^{-1} & O \\ O & O & O & I \end{pmatrix};$$

then we obtain

$$\left(\begin{array}{cc}\begin{pmatrix} O & O & (I_{m_1},O) \\ O & \Sigma & O \\ \begin{pmatrix} I_{n_1} \\ O \end{pmatrix} & O & O \end{pmatrix} & \begin{pmatrix} O & \begin{pmatrix} I_{\tilde{r}_B} & O \\ O & O \end{pmatrix} \\ I_{m_2} & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & I_{n_2} & O \\ \begin{pmatrix} I_{\tilde{r}_C} & O \\ O & O \end{pmatrix} & O & O \end{pmatrix} & O \end{array}\right).$$

The theorem follows by comparing the above with the expressions in Theorem 1. ∎

7. A NUMERICAL EXAMPLE

In this section we give a numerical example to demonstrate the feasibility and accuracy of the algorithm proposed in this paper. All the computation was done on a Sun 3/50 workstation using Matlab version 3.5c. The prescribed structure of Σ_A, Σ_B, and Σ_C is as follows:

$$\Sigma_A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 600 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrices A, B, and C are generated as follows:

$$A = P^{-1}\Sigma_A Q, \qquad B = P^{-1}\Sigma_B U, \qquad C = V^T\Sigma_C Q,$$

TABLE I

OUTPUT FOR A

Columns 1 through 7

940.3657	-548.9146	-59.0867	599.4843	396.5323	-260.5842	1.2036
-384.6252	227.1340	23.0584	-249.2204	-166.5504	109.6126	-0.2079
-95.2420	54.9714	6.7191	-59.2241	-38.9031	25.2749	0.0895
514.9637	-304.2383	-31.7583	333.6676	222.0415	-146.4521	0
-156.2306	93.2910	9.1214	-102.6476	-68.6395	45.3912	0
67.6065	-40.2947	-5.3923	43.8522	29.5923	-18.8303	0
-102.5720	59.4281	7.4395	-64.6689	-42.9251	27.8308	0
-108.6270	64.8006	6.6355	-71.2681	-47.9505	31.4518	0
0.7300	0	0	0	0	0	0
0.3100	0	0	0	0	0	0

Columns 8 through 10

-0.5689	-0.1421	-1.0259
0.1733	-0.3383	-0.2931
0.2424	0.1076	-0.0444
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0

TABLE 2

OUTPUT FOR B

2.2904	-1.1508	1.4127	-0.7208	1.1723	-2.0553	0.0285
-0.4267	0.4626	0.0183	0.2217	0.1785	0.3648	0.8546
-0.2497	0.2244	-0.4270	0.1989	-0.2949	0.9946	-0.0774
1.2102	0.0000	0.0000	0.0000	0.0000	0	0
0.0000	0.8530	0.0000	0.0000	0.0000	0	0
0.0000	0.0000	0.7009	0.0000	0.0000	0	0
0.0000	0.0000	0.0000	0.5746	0.0000	0	0
0.0000	0.0000	0.0000	0.0000	0.2448	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

where P and Q are nonsingular, and U and V are orthogonal. For our particular choice we have

$$\text{Cond}(P)=35.2673, \quad \text{Cond}(Q)=532.1483.$$

After the extraction procedure A is transformed to the form shown in Table 1. The matrix B is transformed to the form shown in Table 2, and the matrix C is transformed to the form shown in Table 3.

TABLE 3

OUTPUT FOR C

Columns 1 through 7

-3.1166	1.4510	0.0562	0.0408	-0.2734	0.1298	0
0.2594	-0.0582	-0.5657	-0.7552	-0.4533	0.1773	0
0.1997	-0.0413	0.8495	-0.5582	-0.1091	0.0399	0
-1.3672	0.1429	-0.1206	-0.3251	0.6218	-0.1328	0
0.4941	-0.0185	0.0110	0.0269	0.1125	0.4309	0
-0.2445	0	0	0	0	0	0
2.2266	0	0	0	0	0	0

Columns 8 through 10

0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0

The regular triplet is

```
(a(4:8,2:6), b(4:8,1:5), c(1:5,2:6)).
```

The computed restricted singular values using the implicit method are

```
1.0e+02 *
  Columns 1 through 4
   0.[illegible]0000000  0.04000000000000  6.00000000000000
0.[illegible]0
  [illegible]
   [illegible].00000000000000
```

The computed ranks of $B_{12}^{(5)}$ and $C_{21}^{(5)}$ are

```
rank(b(1:3,6:7))=2,  rank(c(6:7,1:1))=1.
```

Then we can compute the integer indices as follows:

```
j=2, l=1, k=1, r=3, s1=2, s2=2.
```

The author would like to thank Professors P. Deuflhard and J. Vandewalle for their support of the research. Mrs. S. Wacker's excellent typing of the first version of the manuscript is deeply appreciated. Constructive comments and suggestions of an anonymous referee greatly improved the contents and presentation of the paper.

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Received 11 July 1989; final manuscript accepted 6 March 1991